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Uniform Approximation by Rational Functions Which All Satisfy the Same Algebraic Differential Equation

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We prove that given two disjoint compact sets K_1 and K_2 in the complex plane, without any holes in them, there exists a sequence $p_n(z)$ of rational functions, all of them satisfying one and the same algebraic differential equation, such that $p_n(z)$ converges uniformly to 1 on K_1 and uniformly to 0 on K_2.

There are many theorems in approximation theory, for example the Weierstrass Approximation Theorem, Mergelyan's Theorem, and Runge's Theorem, that assure us of the existence of a sequence (p_n) of rational functions with certain limits. However, these results say nothing about the (differential) complexity of these rational functions. Some attention is now beginning to be paid to whether the p_n can be chosen to be *coherent*, i.e., to all satisfy *one and the same* algebraic differential equation (ADE). From the point of view of the Shannon–Pour-El Thesis (see [SHA], [POE], and [LIR]), this says that there is one general-purpose analog computer that has all of these rational functions p_n as outputs. To produce some particular one, say p_{15000}, you just set the integrators on the computer to certain settings and then let the machine run. A general essay on coherent families of functions can be found in [BOR]. A phrase more illuminating than "coherent" is "uniformly differentially algebraic," which is often used instead, but which takes longer to say and write. For particular connections with approximating families on the real axis, see [BUC], [RUB], [DUF], and [BOS].

Before we state and prove our actual theorem, let it be said that many open questions remain. First, can we choose the p_n to be polynomials in our main theorem? One would hope to "translate the poles" to ∞ in

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a coherent way, but we have not been able to do this. Also, is there is universal ADE that works for all K_1 and K_2? Even in the case of the open unit disc $\mathbb{D}$, is every holomorphic function f that satisfies an ADE the uniform limit on compact subsets of $\mathbb{D}$ of a coherent sequence (p_n) of polynomials? As shown in [BOR], if $f(z) = z/(e^z - 1)$, then $p_n = s_n$, the nth partial sum of the Taylor series of f, is not coherent.

THEOREM. *Let K_1 and K_2 be two disjoint compact sets in $\mathbb{C}$, each having a connected complement. There exists a sequence $(p_n(z))$ of rational functions, which all satisfy one and the same algebraic differential equation, that converges uniformly to* 1 *on K_1 and uniformly to* 0 *on K_2.*

In broad outline, the proof follows the beginning of the proof of Runge's Theorem in [SAZ]. The function to be approximated (1 on K_1 and 0 on K_2) is written as a Cauchy integral. This Cauchy integral is approximated by certain Riemann sums, which are rational functions with simple poles that more or less surround our compact sets. We must do the same thing here, but take care at each step that our rational functions are coherent, i.e., they all satisfy some one ADE. This causes some extra complications which we will now outline.

(1) To get coherence, we have taken the contour of the Cauchy integral to be a lemniscate (actually a single lemniscatic oval). We do this by means of certain Fekete polynomials.

(2) The particular Riemann sums are chosen by using certain Faber polynomials, in order to get coherence.

As a standard application of this theorem, one can easily prove the following "sewing together" theorem.

APPLICATION. *Let $K = \bigcup_{j=1}^{n} K_j$, where the K_j are disjoint compact sets with connected complements. Let $f: K \to \mathbb{C}$ be a function such that for each $j = 1, ..., N$, $f|_{K_j}$ is the uniform limit on K_j of a sequence of rational functions with no poles in K. Then f itself is the uniform limit on K of such a sequence.*

It is not possible to say more of a general nature at this point, so we begin the proof.

LEMMA 1. *Let K_1 and K_2 be two disjoint compact sets in $\mathbb{C}$ whose complements are connected. Then there is a lemniscate L, which is actually a single Jordan curve, that contains K_1 in its inside and K_2 in its outside.*

Proof. For an excellent general treatment of lemniscates, see Chapter 16 of [HIL], which we shall rely on heavily. In particular, we invoke Exercise 16.1.8, a consequence of the Riemann–Macdonald theorem, that the lemniscate $\{|P(z)| = C\}$ (P a polynomial), consists of a single oval (i.e., is a Jordan curve) if and only if $|P(z)| < C$ at all the zeros of $P'(z)$. The polynomials we shall work with are the Fekete polynomials, $F_n(z:K_1) = \prod_{j=1}^{n} (z - z_{n,j})$ (see [HIL, p. 272]), where the $z_{n,j}$, $j = 1, ..., n$ maximize $\prod_{1 \leqslant j < k \leqslant n} |z_j - z_k|$ for the z_j (and z_k) lying in K_1.

This maximum is written as $[\delta_n(K_1)]^{(1/2)\, n(n-1)}$, and it is known that the δ_n decrease to $\rho(K_1)$, the mapping radius (transfinite diameter) of the exterior of K_1. Further, if we let $f_n(z) = [F_n(z:K_1)]^{1/n}$ (principal branch) then $\lim_{n \to \infty} f_n(z) = F(z)$ exists in $G = C(K_1)$, and F maps G conformally onto $D_0 = \{|w| > \rho(K_1)\}$. We shall study the lemniscate

$$L_{n,\varepsilon} = \{F_n(z) = [(1+\varepsilon)\, \rho(K_1)]^n\}$$

and the associated lemniscatic region

$$\Lambda_{n,\varepsilon} = \{|F_n(z)| < [(1+\varepsilon)\, \rho(K_1)]^n\}.$$

Sublemma A. *If $\varepsilon > 0$ is fixed, then there is an $n_0 = n_0(\varepsilon)$ such that K_1 lies inside $L_{n,\varepsilon}$ for all $n \geqslant n_0$.*

Proof. (This also follows from [HIL, Theorem 16.2.3], but we give a different proof.) We have, for $z \in K_1$,

$$|F_n(z:K_1)| = \prod |z - z_{n,j}| = \frac{\prod |z - z_{nj}| \prod |z_{nj} - z_{nk}|}{\prod |z_{nj} - z_{nk}|}$$
$$\leqslant \frac{\delta_{n+1}^{(1/2)(n+1)\, n}}{\delta_n^{(1/2)\, n(n-1)}} \leqslant \frac{\delta_n^{(1/2)(n+1)\, n}}{\delta_n^{(1/2)\, n(n-1)}} = \delta_n^n$$

since (δ_n) decreases with n [HIL, Theorem 16.2.1]. Since $\delta_n \to \rho(K_1)$, the result follows. Now we fix $\varepsilon > 0$ so that for all $n \geqslant n_0(\varepsilon)$ this lemniscatic region $\Lambda_{n,\varepsilon}$ will contain K_1 and exclude K_2. But we want $L_{n,\varepsilon}$ to consist of a single oval. Let

$$\lambda_n(z) = \frac{d}{dz} [F_n(z)]^{1/n}.$$

As mentioned above, it will suffice to show that the zeros of $\lambda_n(z)$ lie, for sufficiently large n, in $\Lambda_{n,\varepsilon}$. Since $f_n(z) = [F_n(z:K_1)]^{1/n}$ converges compactly to $F(z)$ in $C(K_1)$, $f'_n(z) = \lambda_n(z)$ converges compactly there to $F'(z)$. But since F is a conformal map, $F'(z)$ has no zeros on $C(K_1)$.

By Hurwitz' theorem, then, for our fixed ε and n_0 sufficiently large, all the zeros of $\lambda_n(z)$ will lie inside $\Lambda_{n,\varepsilon}$, and the result follows.

In [BOR], it is proved that sums, products, compositions, etc. of coherent families of analytic functions result again in coherent families. We shall use this fact often.

LEMMA 2. *Let γ be the lemniscate of Lemma 1. Then there is one algebraic differential equation $\bar{Q}$ such that if $\varepsilon > 0$, then there exists a rational function $F(z)$ with only simple poles that all lie on γ so that $|F(z) - 1| < \varepsilon$ on K_1 and $|F(z)| < \varepsilon$ on K_2, and $F(z)$ satisfies $\bar{Q}$.*

Proof of Lemma 2. We construct the $F(z)$ by using Faber polynomials, for which an excellent general reference is [MAR, Vol. III]. The salient facts are these. We have a continuum L (the lemniscate of Lemma 1), and we let G be the component of the complement of L that contains ∞. Let ϕ be the conformal mapping function

$$\phi\colon G \to \{|w| > p\}, \qquad \left.\frac{\phi(z)}{z}\right|_{z=\infty} = 1.$$

This function $\phi(z)$ has a Laurent expansion at ∞,

$$\phi(z) = z + \alpha_0 + \frac{\alpha_{-1}}{z} + \frac{\alpha_{-2}}{z^2} + \cdots.$$

Given any integer $n > 0$, $[\phi(z)]^n$ has a Laurent expansion at ∞

$$[\phi(z)]^n = z^n + \alpha_{n-1}^{(n)} z^{n-1} + \cdots + \alpha_0^{(n)}(n) + \frac{\alpha_{-1}^{(n)}}{z} + \frac{\alpha_{-2}^{(n)}}{z^2} + \cdots.$$

The polynomials

$$\phi_n(z) = z^n + \alpha_{n-1}^{(n)} z^{n-1} + \cdots + \alpha_0^{(n)}$$

(achieved by truncating this Laurent expansion) are called the *Faber polynomials* for L. Further, if $\psi(w)$ is the inverse of $\phi(z)$, then we have

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{\phi_n(z)}{w^{n+1}}.$$

Finally, if L is the lemniscate (with k foci)

$$L = \{|z^k + A_{k-1} z^{k-1} + \cdots + A_0| = \tilde{\rho}^k\} \qquad (*)$$

where $\tilde{\rho}$ is chosen so that L is still an oval that surrounds K_1, and excludes K_2, then

$$\phi(z) = z\left(1 + \frac{A_{k-1}}{z} + \cdots + \frac{A_0}{z^k}\right)^{1/k}$$

and hence, for $m = 0, 1, 2, ...,$

$$\phi_{mk}(z) = (z^k + A_{k-1} z^{k-1} + \cdots + A_0)^m.$$

We are supposing that our lemniscate is given by $(*)$, where k is the number of foci. Also for convenience, we will suppose that $\tilde{\rho} = 1$.

We choose n to be a large integer and let

$$F(z) = \frac{1}{2\pi i} \sum_{kl<n} \frac{\psi'(\omega^{kl})}{\psi(\omega^{kl}) - z} (\omega^{k(l+1)}) - \omega^{kl}),$$

where $\omega = \exp(2\pi i/n)$ is a primitive nth root of unity. But this is just an approximating Riemann sum for

$$\frac{1}{2\pi i} \int_{|w|=1} \frac{\psi'(w)}{\psi(w) - z} \, dw = \frac{1}{2\pi i} \int_L \frac{d\xi}{\xi - z}.$$

Now $1/2\pi i \int_L d\xi/(\xi - z)$ equals 1 on K_1, and 0 on K_2. Hence for n large enough, we have the estimates $|F(z) - 1| < \varepsilon$ on K_1, and $|F(z)| < \varepsilon$ on K_2 as required.

The harder part is to make the $F(z)$ (which depend on n) satisfy some one $\bar{Q}$ (which is independent of n). We have

$$F(z) = \frac{(\omega^k - 1)}{2\pi i} \sum_{kl<n} \frac{\psi'(\omega^{kl})}{\psi(\omega^{kl}) - z} \omega^{kl}.$$

Since all constants form a coherent family, we need only work with

$$\begin{aligned} F_1(z) &= \sum_{l<n/k} \frac{\psi'(\omega^{kl})}{\psi(\omega^{kl}) - z} \omega^{kl} \\ &= \sum_{l<n/k} \sum_{r=0}^{\infty} \frac{\phi_r(z)}{\omega^{rkl}} = \sum_{r=0}^{\infty} \phi_r(z) \left[\sum_{l<n/k} \omega^{-rkl} \right]. \end{aligned}$$

Restrict n to be a (large) multiple of k. Now suppose $kr \neq 0 \bmod n$. Then the inner sum (in brackets) is

$$\frac{1 - \omega^{-rn}}{1 - \omega^{-kr}},$$

which vanishes since ω is an nth root of unity. Now if $kr \equiv 0 \bmod n$, then the summand ω^{-rkl} equals 1, and the inner sum becomes n/k, and we get

$$F_1(z) = (n/k) \sum_{kr \equiv 0 \bmod n} \phi_r(z).$$

We now let $n = uk^2$, where u is a large integer. Then $kr \equiv 0 \bmod n$ if and only if $r = ukm$ for some positive integer m. Then

$$\begin{aligned} F_1(z) &= (n/k) \sum_{m=0}^{\infty} \phi_{kum}(z) \\ &= (n/k) \sum_{m=0}^{\infty} [z^k + A_{k-1} z^{k-1} + \cdots + A_0]^{um} \\ &= (n/k) \frac{1}{1 - [z^k + A_{k-1} z^{k-1} + \cdots + A_0]^u}, \end{aligned}$$

which certainly forms a coherent family since the family

$$\left\{ C \frac{1}{1 - z^s} : C \in \mathbb{C},\ s = 1, 2, 3, ... \right\}$$

is coherent, because $[z^k + A_{k-1} z^{k-1} + \cdots + A_0]$ is a polynomial, and because compositions of coherent families are coherent. This proves the Lemma.

The theorem is proved.

References

[BOR] M. Boshernitzan and L. A. Rubel, Coherent families of polynomials, *Analysis* **6** (1986), 339–389.

[BOS] M. Boshernitzan, Universal formulae and universal differential equations, *Ann. of Math.* **124** (1986), 273–291.

[BUC] R. C. Buck, The solutions to a smooth PDE can be dense in $C[I]$, *J. Differential Equations* **41** (1981), 239–244.

[DUF] R. J. Duffin, Rubel's universal differential equation, *Proc. Nat. Acad. Sci. U.S.A.* **78** (1981), 4661–4662.

[HIL] E. Hille, "Analytic Function Theory," Ginn and Co., Boston, 1959–1962.

[LIR] L. Lipshitz and L. A. Rubel, A differentially algebraic replacement theorem and analog computability, *Proc. Amer. Math. Soc.* **99** (1987), 367–372.

[MAR] A. I. Markushevich, "Theory of Functions of a Complex Variable," Prentice–Hall, Englewood Cliffs, NJ, 1965.

[POE] M. B. Pour-El, Abstract computability and its relation to the general-purpose analog computer, etc., *Trans. Amer. Math. Soc.* **199** (1974), 1–29.

[RUB] L. A. Rubel, A universal differential equation, *Bull. Amer. Math. Soc. (N.S.)* **4** (1981), 345–349.

[SHA] C. E. Shannon, Mathematical theory of the differential analyzer, *J. Math. Phys. Mass. Inst. Tech.* **20** (1941), 337–354.

[SAZ] S. Saks and A. Zygmund, "Analytic Functions," Monografie Matematyczne, Warsaw, 1952.